ON p-ADIC GIBBS MEASURES OF COUNTABLE STATE POTTS MODEL ON THE CAYLEY TREE

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Abstract. In this paper we consider countable state p-adic Potts model on the Cayley tree. A construction of p-adic Gibbs measures which depends on weights λ is given, and an investigation of such measures is reduced to examination of an infinite-dimensional recursion equation. Studying the derived equation under some condition concerning weights, we prove the absence of a phase transition. Note that the condition does not depend on values of the prime p, and an analogues fact is not true when the number of spins is finite. For homogeneous model it is shown that the recursive equation has only one solution under that condition on weights. This means that there is only one p-adic Gibbs measure μ_λ. The boundedness of the measure is also established. Moreover, continuous dependence of the measure μ_λ on λ is proved. At the end we formulate one limit theorem for μ_λ.

Mathematics Subject Classification: 46S10, 82B26, 12J12, 39A70, 47H10, 60K35.
Key words: p-adic numbers, countable state Potts model, Gibbs measure, weight, uniqueness, boundednes.

1. INTRODUCTION

Since the 1980s, various models described in the language of p-adic analysis have been actively studied [5],[22, 23],[47],[62]. More precisely, models defined over the field of p-adic numbers have been considered. Which is due to the assumption that p-adic numbers provide a more exact and more adequate description of microworld phenomena. The well-known studies in this area are primarily devoted to investigating quantum mechanics models using equations of mathematical physics [4, 48, 52, 63, 61]. One of the first applications of p-adic numbers in quantum physics appeared in the framework of quantum logic in [10]. This model is especially interesting for us because it could not be described by using conventional real valued probability. Furthermore, numerous applications of the p-adic analysis to mathematical physics have been proposed in [8],[32],[33],[52]. Besides, it is also known [33, 42, 47, 52, 54, 60, 61] that a number of p-adic models in physics cannot be described using ordinary Kolmogorov's probability theory. New probability models, namely p-adic ones were investigated in [15],[17],[31],[40]. After that in [41] an abstract p-adic probability theory was developed by means of the theory of non-Archimedean measures [54]. Using that measure theory in [39],[45] the theory of stochastic processes with values in p-adic and more general non-Archimedean fields having probability distributions with non-Archimedean values has been developed. In particular, a non-Archimedean analog of the Kolmogorov theorem was proven (see also [25]). Such a result allows us to construct wide classes of stochastic processes using finite dimensional probability distributions. We point out that stochastic processes on the field $\mathbb{Q}_p$ of p-adic numbers have been studied by many authors, for

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example, [1, 2, 3, 18, 43, 64]. In those investigations wide classes of Markov processes on $\mathbb{Q}_p$ were constructed and studied. Such studies, therefore, give a possibility to develop the theory of statistical mechanics in the context of the p-adic theory, since it lies on the basis of the theory of probability and stochastic processes. Note that one of the central problems of such a theory is the study of infinite-volume Gibbs measures corresponding to a given Hamiltonian, and a description of the set of such measures. In most cases such an analysis depend on a specific properties of Hamiltonian, and complete description is often a difficult problem. This problem, in particular, relates to a phase transition of the model (see [27]).

In [36, 37] a notion of ultrametric Markovianity, which describes independence of contributions to random field from different ultrametric balls, has been introduced, and shows that Gaussian random fields on general ultrametric spaces (which were related with hierarchical trees), which were defined as a solution of pseudodifferential stochastic equation (see also [29]), satisfies the Markovianity. In addition, covariation of the defined random field was computed with the help of wavelet analysis on ultrametric spaces (see also [44]). Some applications of the results to replica matrices, related to general ultrametric spaces have been investigated in [38].

In this paper we develop a p-adic probability theory approach to study countable state of nearest-neighbor Potts models on a Cayley tree (see [9]) over p-adic filed. We are especially interested in the construction of p-adic Gibbs measure for the mentioned model. Such measures present more natural concrete examples of p-adic Markov processes (see [37, 39], for definitions). When states are finite, say q, then the corresponding p-adic q-state Potts models on the same tree have been studied in [49, 50, 51][1]. It was established that a phase transition occurs if q is divisible by p. This shows that the transition depends on the number of spins q. To establish such a result we investigated fixed points of a p-adic dynamical systems associated with that model. We remark that first investigations of non-Archimedean dynamical systems have appeared in [28]. We also point out that intensive development of p-adic (and more general algebraic) dynamical systems has happened few years, see [6, 7, 11, 14, 40, 57, 59, 65]. More extensive lists may be found in the p-adic dynamics bibliography maintained by Silverman [57] and the algebraic dynamics bibliography of Vivaldi [60].

The aim of this paper is to give sufficient condition for the uniqueness of p-adic Gibbs measures of the countable state Potts model, and to study such measures. Note that in comparison to a real case, in a p-adic setting, à priori the existence of such kind of measures for the model is not known, since there is not much information on topological properties of the set of all p-adic measures defined even on compact spaces. However, in the real case, there is the so called the Dobrushin's Theorem [19, 20] which gives a sufficient condition for the existence of the Gibbs measure for a large class of Hamiltonians.

The paper is organized as follows. After preliminaries (Sec. 2) in section 3 we define our model, and give a construction of p-adic Gibbs measures which depends on weight λ. Using Kolmogorov extension Theorem [39], an investigation of such measures is reduced to examination of an infinite-dimensional recursion equation. In Section 4, studying the derived equation under some condition on weights, we prove the absence of a phase transition. Note that, for the real counterparts of the model, such results are unknown (see [24, 26]). It turns out that the founding condition does not depend on values of the prime p, and therefore, an analogous fact is not true when the number of spins is finite. In section 5, we well consider homogeneous p-adic Potts model, and show under the condition formulated in section 3, the recursive equation has only one solution. Hence, there is only one p-adic Gibbs measure μ_λ.

[1] The classical (real value) counterparts of such models were considered in [26], [66]

Then we establish boundedness one. In section 6, we prove continuous dependence of the measure μ_λ on λ. We also prove one limit theorem for μ_λ. The last section is devoted to the conclusions of the results.

2. Preliminaries

In what follows p will be a fixed prime number, and $\mathbb{Q}_p$ denotes the field of p-adic filed, formed by completing $\mathbb{Q}$ with respect to the unique absolute value satisfying $|p|_p = 1/p$. The absolute value $|\cdot|_p$, is non- Archimedean, meaning that it satisfies the ultrametric triangle inequality $|x+y|_p \le \max\{|x|_p, |y|_p\}$.

Given $a \in \mathbb{Q}_p$ and $r > 0$ put

$$B(a,r) = \{x \in \mathbb{Q}_p : |x-a|_p < r\}.$$

The *p-adic logarithm* is defined by the series

$$\log_p(x) = \log_p(1+(x-1)) = \sum_{n=1}^{\infty}(-1)^{n+1}\frac{(x-1)^n}{n},$$

which converges for $x \in B(1,1)$; the *p-adic exponential* is defined by

$$\exp_p(x) = \sum_{n=0}^{\infty}\frac{x^n}{n!},$$

which converges for $x \in B(0, p^{-1/(p-1)})$.

Lemma 2.1. [42, 55] *Let $x \in B(0, p^{-1/(p-1)})$ then we have*

$$|\exp_p(x)|_p = 1, \quad |\exp_p(x)-1|_p = |x|_p, \quad |\log_p(1+x)|_p = |x|_p \tag{2.1}$$

$$\log_p(\exp_p(x)) = x, \quad \exp_p(\log_p(1+x)) = 1+x. \tag{2.2}$$

Note the basics of p-adic analysis, p-adic mathematical physics are explained in [42, 55, 61].

Let $(X, \mathcal{B})$ be a measurable space, where $\mathcal{B}$ is an algebra of subsets X. A function $\mu : \mathcal{B} \to \mathbb{Q}_p$ is said to be a *p-adic measure* if for any $A_1, \dots, A_n \subset \mathcal{B}$ such that $A_i \cap A_j = \emptyset$ $(i \ne j)$ the equality holds

$$\mu\bigg(\bigcup_{j=1}^{n} A_j\bigg) = \sum_{j=1}^{n}\mu(A_j).$$

A p-adic measure is called a *probability measure* if $\mu(X) = 1$. A p-adic probability measure μ is called *bounded* if $\sup\{|\mu(A)|_p : A \in \mathcal{B}\} < \infty$. Note that in general, a p-adic probability measure need not be bounded [31, 39, 42]. For more detail information about p-adic measures we refer to [31],[40],[54].

Recall that the Cayley tree (see [9]) Γ^k of order $k \ge 1$ is an infinite tree, i.e., a graph without cycles, from each vertex of which exactly $k+1$ edges issue. Let $\Gamma^k = (V, L)$, where V is the set of vertexes of Γ^k, L is the set of edges of Γ^k. The vertices x and y are called *nearest neighbors* and they are denoted by $l = <x, y>$ if there exists an edge connecting them. A collection of the pairs $<x, x_1>, \dots, <x_{d-1}, y>$ is called a *path* from the point x to the point y. The distance $d(x,y), x, y \in V$, on the Cayley tree, is the length of the shortest path from x to y. Now fix $x^0 \in V$, and set

$$W_n = \{x \in V | d(x, x^0) = n\}, \quad V_n = \bigcup_{m=1}^{n} W_m, \quad L_n = \{l = <x,y> \in L | x, y \in V_n\}.$$

The set of *direct successors* of x is defined by

$$S(x) = \{y \in W_{n+1} : d(x,y) = 1\}, \quad x \in W_n. \tag{2.3}$$

Observe that any vertex $x \neq x^0$ has k direct successors and x^0 has $k+1$.

3. The p-adic Potts model and p-adic Gibbs measures

We consider the p-adic Potts model where spin takes values in the set $\Phi = \{0, 1, 2, \cdots\}$ (Φ is called a *state space*) and is assigned to the vertices of the tree $\Gamma^k = (V, \Lambda)$. A configuration σ on V is then defined as a function $x \in V \to \sigma(x) \in \Phi$; in a similar manner one defines configurations σ_n and ω on V_n and W_n, respectively. The set of all configurations on V (resp. V_n, W_n) coincides with $\Omega = \Phi^V$ (resp. $\Omega_{V_n} = \Phi^{V_n}$, $\Omega_{W_n} = \Phi^{W_n}$). One can see that $\Omega_{V_n} = \Omega_{V_{n-1}} \times \Omega_{W_n}$. Using this, for given configurations $\sigma_{n-1} \in \Omega_{V_{n-1}}$ and $\omega \in \Omega_{W_n}$ we define their concatenations by

$$(\sigma_{n-1} \vee \omega)(x) = \begin{cases} \sigma_{n-1}(x), & \text{if } x \in V_{n-1}, \\ \omega(x), & \text{if } x \in W_n. \end{cases}$$

It is clear that $\sigma_{n-1} \vee \omega \in \Omega_{V_n}$.

The Hamiltonian $H_n : \Omega_{V_n} \to \mathbb{Q}_p$ of the inhomogeneous p-adic countable state Potts model has a form

$$H_n(\sigma) = \sum_{<x,y>\in L_n} J_{x,y}\delta_{\sigma(x),\sigma(y)}, \quad \sigma \in \Omega_{V_n}, \ n \in \mathbb{N}, \tag{3.1}$$

where δ is the Kronecker symbol and the coupling constants $J_{x,y}$ are taken from $\mathbb{Q}_p$ with constraint

$$|J_{x,y}|_p < \frac{1}{p^{1/(p-1)}}, \quad \forall < x, y >\in L_n. \tag{3.2}$$

Note that such a condition provides the existence of a p-adic Gibbs measure (see (3.5)).

We say (3.1) is *homogeneous Potts model* if $J_{xy} = J, \ \forall < x, y >$.

We then construct p-adic Gibbs measures corresponding to the model.

A given set A we put $\mathbb{Q}_p^A = \{\{x_i\}_{i\in A} : x_i \in \mathbb{Q}_p\}$.

Assume that a function $\mathbf{h} : V \setminus \{x^{(0)}\} \to \mathbb{Q}_p^\Phi$, i.e. $\mathbf{h}_x = \{h_{i,x}\}_{i\in\Phi}$, is such that

$$|h_{i,x}|_p < \frac{1}{p^{1/(p-1)}} \quad \text{for all} \ \ x \in V \setminus \{x^{(0)}\}, \ i \in \Phi, \tag{3.3}$$

and a non-zero element $\lambda = \{\lambda(i)\}_{i\in\Phi} \in \mathbb{Q}_p^\Phi$ is fixed such that

$$|\lambda(n)|_p \to 0 \quad \text{as} \quad n \to \infty \tag{3.4}$$

which is called a *weight*. In what follows, without losing generality we may assume that $\lambda(0) \neq 0$.

Given $n = 1, 2, \ldots$ a p-adic probability measure $\mu_{\mathbf{h}}^{(n)}$ on Ω_{V_n} is defined by

$$\mu_{\mathbf{h}}^{(n)}(\sigma) = \frac{1}{Z_n^{(\mathbf{h})}} \exp_p\left\{H_n(\sigma) + \sum_{x\in W_n} h_{\sigma(x),x}\right\} \prod_{x\in V_n} \lambda(\sigma(x)) \tag{3.5}$$

Here, $\sigma \in \Omega_{V_n}$, and $Z_n^{(\mathbf{h})}$ is the corresponding normalizing factor called a *partition function* given by

$$Z_n^{(\mathbf{h})} = \sum_{\sigma\in\Omega_{V_n}} \exp_p\left\{H_n(\sigma) + \sum_{x\in W_n} h_{\sigma(x),x}\right\} \prod_{x\in V_n} \lambda(\sigma(x)), \tag{3.6}$$

here subscript n and superscript $(\mathbf{h})$ are accorded to the Z, since it depends on n and a function $\mathbf{h}$.

The conditions (3.2) and (3.3) allow the existence of $\exp_p$, therefore, the measures $\mu_{\mathfrak{h}}^{(n)}$ are well defined.

Now we want to define a p-adic probability measure μ on Ω such that it would be compatible with defined ones $\mu_{\mathbf{h}}^{(n)}$, i.e.

$$\mu(\sigma \in \Omega : \sigma|_{V_n} = \sigma_n) = \mu_{\mathbf{h}}^{(n)}(\sigma_n), \quad \text{for all } \ \sigma_n \in \Omega_{V_n}, \ n \in \mathbb{N}. \tag{3.7}$$

In general, à priori the existence of such kind of measure μ is not known, since, there is not much information on topological properties, such as compactness, of the set of all p-adic measures defined even on compact spaces[2]. Therefore, at a moment, we can only use the so called *compatibility condition* for the measures $\mu_{\mathbf{h}}^{(n)}$, $n \geq 1$, i.e.

$$\sum_{\omega \in \Omega_{W_n}} \mu_{\mathbf{h}}^{(n)}(\sigma_{n-1} \vee \omega) = \mu_{\mathbf{h}}^{(n-1)}(\sigma_{n-1}), \tag{3.8}$$

for any $\sigma_{n-1} \in \Omega_{V_{n-1}}$ (cp. [16]), which implies the existence of a unique p-adic measure μ defined on Ω with a required condition (3.7). Moreover, if the measures $\mu_{\mathbf{h}}^{(n)}$ are bounded, then μ is also bounded. This assertion is known as the p-adic Kolmogorov extension Theorem (see [25],[39]).

So, if for some function $\mathbf{h}$ the measures $\mu_{\mathbf{h}}^{(n)}$ satisfy the compatibility condition, then there is a unique p-adic probability measure, which we denote by $\mu_{\mathbf{h}}$, since it depends on $\mathbf{h}$. Such a measure $\mu_{\mathbf{h}}$ is said to be *p-adic Gibbs measure* corresponding to the p-adic Potts model. By $\mathcal{S}$ we denote the set of all such p-adic Gibbs measures. If there are at least two different p-adic Gibbs measures in $\mathcal{S}$, i.e. one can find two different functions $\mathbf{s}$ and $\mathbf{h}$ defined on $V \setminus \{x^0\}$ such that there exist the corresponding measures $\mu_{\mathbf{s}}$ and $\mu_{\mathbf{h}}$, which are different, then we say that *a phase transition* occurs for the model, otherwise, there is *no phase transition.*

Now one can ask for what kind of functions $\mathbf{h}$ the measures $\mu_{\mathbf{h}}^{(n)}$ defined by (3.5) would satisfy the compatibility condition (3.8). The following theorem gives an answer to this question.

Theorem 3.1. *The measures $\mu_{\mathbf{h}}^{(n)}$, $n = 1, 2, \dots$ given by* (3.5)*, satisfy the compatibility condition* (3.8) *if and only if for any $x \in V \setminus \{x^{(0)}\}$ the following equation holds:*

$$\hat{h}_{i,x} = \frac{\lambda(i)}{\lambda(0)} \prod_{y \in S(x)} F_i(\hat{\mathbf{h}}_y; \theta_{x,y}), \quad i \in \mathbb{N} \tag{3.9}$$

here and below $\theta_{x,y} = \exp_p(J_{x,y})$, a sequence $\hat{\mathbf{h}} = \{\hat{h}_i\}_{i \in \mathbb{N}} \in \mathbb{Q}_p^{\mathbb{N}}$ is defined by $\mathbf{h} = \{h_i\}_{i \in \Phi}$ as follows

$$\hat{h}_i = \exp_p(h_i - h_0) \frac{\lambda(i)}{\lambda(0)}, \quad i \in \mathbb{N} \tag{3.10}$$

and mappings $F_i : \mathbb{Q}_p^{\mathbb{N}} \times \mathbb{Q}_p \to \mathbb{Q}_p$ are defined by

$$F_i(\mathbf{x}; \theta) = \frac{(\theta - 1)x_i + \sum_{j=1}^{\infty} x_j + 1}{\sum_{j=1}^{\infty} x_j + \theta}, \quad \mathbf{x} = \{x_i\}_{i \in \mathbb{N}}, \ \theta \in \mathbb{Q}_p, \ i \in \mathbb{N}. \tag{3.11}$$

The proof consists of checking condition (3.8) for the measures (3.5) (cp. [26]).

[2]In the real case, when the state space is compact, then the existence follows from the compactness of the set of all probability measures (i.e. Prohorov's Theorem). When the state space is non-compact, then there is a Dobrushin's Theorem [19, 20] which gives a sufficient condition for the existence of the Gibbs measure for a large class of Hamiltonians. In [24] using that theorem it has been established the existence of the Gibbs measure for the real counterpart of the studied Potts model. It should be noted that there are even nearest-neighbor models with countable state space for which the Gibbs measure does not exists [58].

Remark 3.1. Note that thanks to the non-Archimedeanity of the norm $|\cdot|_p$ the series $\sum_{k=1}^{\infty} x_k$ converges iff the sequence $\{x_n\}$ converges to 0 (see [42, 55]). Therefore, from (3.10) one can see that condition (3.4) and $|\exp_p(x)|_p = 1$ imply that the series $\sum_{j=1}^{\infty} \hat{h}_{j,x}$ always converges and is finite.

4. Absence of the phase transition

In this section, under some condition on weights λ, we are going to prove the absence of the phase transition for the model (3.1)

As pointed out from (3.10),(3.4) we have $|\hat{h}_n|_p \to 0$ as $n \to \infty$. Therefore, let us consider the following space

$$c_0 = \{\mathbf{x} = \{x_n\}_{n\in\mathbb{N}} \in \mathbb{Q}_p^{\mathbb{N}} : \ |x_n|_p \to 0, \ n \to \infty\}$$

with a norm $\|\mathbf{x}\| = \max_n |x_n|_p$ (see [54, 55] for more p-adic Banach spaces). Put $\mathbf{B} = \{\mathbf{x} \in c_0 : \ \|\mathbf{x}\| < 1\}$. Since the norm takes discrete values, the set $\mathbf{B}$ coincides with $\mathbf{B} = \{\mathbf{x} \in c_0 : \ \|\mathbf{x}\| \leq 1/p\}$. Therefore, $\mathbf{B}$ is a closed set.

Let us for the sake of shortness, given a sequence $\mathbf{x} = \{x_n\}_{n\in\mathbb{N}}$, denote

$$X := \sum_{j=1}^{\infty} x_j. \tag{4.1}$$

Then one can easily see that

$$|X - Y|_p = \left|\sum_{j=1}^{\infty}(x_j - y_j)\right|_p \leq \max_j |x_j - y_j| = \|\mathbf{x} - \mathbf{y}\|, \tag{4.2}$$

for any $\mathbf{x}, \mathbf{y} \in c_0$.

Lemma 4.1. *For the mapping F_i given by* (3.11) *the following relations hold*

$$|F_i(\mathbf{x}, \theta) - F_i(\mathbf{y}, \theta)|_p \leq |\theta - 1|_p \|\mathbf{x} - \mathbf{y}\|, \tag{4.3}$$

$$|F_i(\mathbf{x}, \theta) - 1|_p = |\theta - 1|_p. \tag{4.4}$$

for every $\mathbf{x}, \mathbf{y} \in \mathbf{B}$ and $i \in \mathbb{N}$.

Proof. Let $\mathbf{x}, \mathbf{y} \in \mathbf{B}$, then from (3.11) with (4.2) we have

$$\begin{aligned}
|F_i(\mathbf{x}, \theta) - F_i(\mathbf{y}, \theta)|_p &= |((\theta - 1)x_i + X + 1)(Y + \theta) - ((\theta - 1)y_i + Y + 1)(X + \theta)|_p \\
&= |\theta - 1|_p |x_i Y - y_i X + \theta(x_i - y_i) + X - Y|_p \\
&= |\theta - 1|_p \Big|(X + \theta)(x_i - y_i) + (1 - x_i)(X - Y)|_p \\
&\leq |\theta - 1|_p \max\{|x_i - y_i|_p, |X - Y|_p\} \\
&\leq |\theta - 1|_p \|\mathbf{x} - \mathbf{y}\|
\end{aligned}$$

which proves (4.3), here we have used that $|X + \theta|_p = 1$,$|Y + \theta|_p = 1$. Next relation (4.4) is obvious. □

Let us first enumerate $S(x)$ for any $x \in V$ as follows $S(x) = \{x_1, \cdots, x_k\}$, here as before $S(x)$ is the set of direct successors of x (see (2.3)). Using this enumeration one can rewrite (3.9) by

$$\hat{h}_{i,x} = \frac{\lambda(i)}{\lambda(0)} \prod_{m=1}^{k} F_i(\hat{\mathbf{h}}_{x_m}; \theta_{x,x_m}), \quad i \in \mathbb{N}, \quad \text{for every} \ \ x \in V \setminus \{x^{(0)}\}. \tag{4.5}$$

Now we need an auxiliary fact.

Lemma 4.2. *If* $|a_i|_p \leq 1$, $|b_i|_p \leq 1$, $i = 1, \dots, n$, *then*

$$\left|\prod_{i=1}^{n} a_i - \prod_{i=1}^{n} b_i\right|_p \leq \max_{i \leq i \leq n}\{|a_i - b_i|_p\}$$

Proof. We have

$$\begin{aligned}
\left|\prod_{i=1}^{n} a_i - \prod_{i=1}^{n} b_i\right|_p &\leq \left|a_1\left(\prod_{i=2}^{n} a_i - \prod_{i=2}^{n} b_i\right) + (a_1 - b_1)\prod_{i=2}^{n} b_i\right|_p \\
&\leq \max\left\{|a_1 - b_1|_p, \left|\prod_{i=2}^{n} a_i - \prod_{i=2}^{n} b_i\right|_p\right\} \\
&\leq \cdots \\
&\leq \max_{i \leq i \leq n}\{|a_i - b_i|_p\}
\end{aligned}$$

which is the assertion. □

So, we can formulate the main result.

Theorem 4.3. *Assume that a weight* λ *satisfies the following condition*

$$\max_i \left|\frac{\lambda(i)}{\lambda(0)}\right|_p < 1. \tag{4.6}$$

Then there is no phase transition for the countable state p*-adic Potts model* (3.1), *i.e.* $|\mathcal{S}| \leq 1$.

Proof. From (4.6) and (3.10) we see that all solutions of (4.5) belong to $\mathbf{B}$. Now let us assume that $\hat{\mathbf{h}} = \{\hat{\mathbf{h}}_x, x \in V \setminus \{x^{(0)}\}\}, \hat{\mathbf{s}} = \{\hat{\mathbf{s}}_x, x \in V \setminus \{x^{(0)}\}\}$ be the two solutions of (4.5). Now fix an arbitrary vertex $x \in V \setminus \{x^{(0)}\}$. Then (4.5) with Lemmas 4.1 and 4.2, implies that

$$\begin{aligned}
|\hat{h}_{i,x} - \hat{s}_{i,x}|_p &= \left|\frac{\lambda(i)}{\lambda(0)}\right|_p \left|\prod_{m=1}^{k} F_i(\hat{\mathbf{h}}_{x_m}; \theta_{x,x_m}) - \prod_{m=1}^{k} F_i(\hat{\mathbf{s}}_{x_m}; \theta_{x,x_m})\right|_p \\
&\leq \left|\frac{\lambda(i)}{\lambda(0)}\right|_p \max_{1 \leq m \leq k}\left\{|F_i(\hat{\mathbf{h}}_{x_m}; \theta_{x,x_m}))_i - F_i(\hat{\mathbf{s}}_{x_m}; \theta_{x,x_m})|_p\right\} \\
&\leq \max_{1 \leq m \leq k}\left\{|\theta_{x,x_m} - 1|_p \|\hat{\mathbf{h}}_{x_m} - \hat{\mathbf{s}}_{x_m}\|\right\} \\
&\leq \frac{1}{p}\max_{1 \leq m \leq k}\left\{\|\hat{\mathbf{h}}_{x_m} - \hat{\mathbf{s}}_{x_m}\|\right\}.
\end{aligned}$$

Hence,

$$\|\hat{\mathbf{h}}_x - \hat{\mathbf{s}}_x\| \leq \frac{1}{p}\max_{1 \leq m \leq k}\left\{\|\hat{\mathbf{h}}_{x_m} - \hat{\mathbf{s}}_{x_m}\|\right\} \tag{4.7}$$

Now take an arbitrary $\epsilon > 0$ and $n_0 \in \mathbb{N}$ such that $1/p^{n_0} < \epsilon$. Iterating (4.7) n_0 times one gets $\|\hat{\mathbf{h}}_x - \hat{\mathbf{s}}_x\| < \epsilon$. Therefore, the arbitrariness of ϵ and x yield that $\hat{\mathbf{h}}_x = \hat{\mathbf{s}}_x$ for every $x \in V \setminus \{x^{(0)}\}$. This means that $|\mathcal{S}| \leq 1$ and completes the proof. □

Remark 4.1. It is clear that the condition (4.6) does not depend on values of the prime p, therefore an analogues fact is not true when the number of spins is finite (see also Remark 5.1 (a)).

Remark 4.2. Note that the equality can be interpreted as an infinite dimensional recurrence equation over the tree. So, Theorem 4.3 means that the equation has no more one solution. Simpler, recurrence equations over p-adic numbers were considered in [21],[48].

5. Uniqueness and boundedness of the Gibbs measure

As we pointed out, in general, p-adic Gibbs measures may not exist. In this section we will show that the p-adic Gibbs measure is unique under the condition (4.6) for the homogeneous model (3.1).

Throughout this section we suppose that $J_{x,y} = J$. Recall that a function $\mathbf{h} = \{\mathbf{h}_x, x \in V \setminus \{x^0\}\}$ is *translation-invariant* if $\mathbf{h}_x = \mathbf{h}_y$ for every $x, y \in V \setminus \{x^0\}$. We are going to show that (3.9) has a translation invariant solution. To this end, consider the following mapping $\mathcal{F} : c_0 \to \mathbb{Q}_p^{\mathbb{N}}$ defined by

$$(\mathcal{F}(\mathbf{x}))_i = \frac{\lambda(i)}{\lambda(0)}(F_i(\mathbf{x}; \theta))^k, \quad i \in \mathbb{N}, \tag{5.1}$$

where $\mathbf{x} = \{x_n\} \in c_0$. One can see that the domain of the mapping is not whole space c_0[3], therefore, in general, it is unbounded. But we are are going to examine $\mathcal{F}$ on $\mathbf{B}$. Basically, for the mapping we do not have the inclusion $\mathcal{F}(\mathbf{B}) \subset \mathbf{B}$. However, from Lemma 4.1 and (4.6) we derive the following

Lemma 5.1. *Let for a weight λ condition* (4.6) *be satisfied. Then $\mathcal{F}(\mathbf{B}) \subset \mathbf{B}$. Moreover,*

$$\|\mathcal{F}(\mathbf{x}) - \mathcal{F}(\mathbf{y})\| \leq |\theta - 1|_p \|\mathbf{x} - \mathbf{y}\| \ \ \textit{for every} \ \ \mathbf{x}, \mathbf{y} \in \mathbf{B}. \tag{5.2}$$

Now noting that $|\theta - 1|_p < 1/p^{1/(p-1)}$ and according to the above Lemma, we can apply the fixed point theorem to $\mathcal{F}$, which implies the existence of a unique fixed point $\hat{\mathbf{h}}_\lambda = \{\hat{h}_{\lambda,n}\}_{n \in \mathbb{N}} \in \mathbf{B}$ (here the solution depends on a weight λ, therefore, we indicate that dependence by subscript λ). Since $\hat{\mathbf{h}}_\lambda$ is a fixed point of $\mathcal{F}$ one has

$$\frac{\lambda(0)}{\lambda(i)}\hat{h}_{\lambda,i} = (F_i(\hat{\mathbf{h}}_\lambda; \theta))^k, \tag{5.3}$$

which, thanks to (4.4) and Lemma 4.2, implies that

$$\left|\frac{\lambda(0)}{\lambda(i)}\hat{h}_{\lambda,i}\right|_p = 1, \qquad \left|\frac{\lambda(0)}{\lambda(i)}\hat{h}_{\lambda,i} - 1\right|_p \leq |\theta - 1|_p$$

Therefore, Lemma 2.1 allows us to take logarithm from both sides of (3.10), and we obtain

$$h_{\lambda,i} - h_{\lambda,0} = \log_p\left(\frac{\lambda(0)}{\lambda(i)}\hat{h}_{\lambda,i}\right),$$

which, due to Theorem 3.1, defines the p-adic Gibbs measure, which is denoted by μ_λ. Now combining this with Theorem 4.3 we have the following

Theorem 5.2. *Let $0 < |J|_p < 1/p^{1/(p-1)}$ and for a weight λ condition* (4.6) *be satisfied. Then for homogeneous p-adic Potts model* (3.1) *on the Cayley tree of order k there is a unique p-adic Gibbs measure μ_λ.*

Remark 5.1. It is worth to emphasize the following notes:

(a) Note that in [50] we have proven for the q-state Potts model that the p-adic Gibbs measure is unique if q and p are relatively prime. Therefore, the proven Theorem 4.3 shows the difference between finite and countable state Potts models.

[3]More exactly, the domain is $\{\mathbf{x} \in c_0 : \sum_{n=1}^{\infty} x_n + \theta \neq 0\}$

(b) It turns out that condition (4.6) is important. If we replace strict inequality there with weaker one $\leq$, then Theorem 4.3 may not hold. Namely, in that case there may occur a phase transition. Indeed, if $\lambda(0) = \lambda(1) = \lambda(2) = 1$ and $\lambda(k) = 0$ for every $k \geq 3$, then clearly (4.6) is not satisfied. On the other hand, our model is reduced to 3-state Potts model. For such a model in [49] the existence of the phase transition was proven at $p = 3$. Moreover, in that case, p-adic Gibbs measures were unbounded.

(c) For the real counterpart of the model uniqueness result is still unknown (see [24, 26]).

Now we propose the following

Problem. Investigate all fixed points and behavior around such points of the dynamical system (5.1) on whole space c_0. Note that the dynamical system is rational. Therefore, we hope that general theory of rational dynamical systems developed in [11, 12, 13, 14, 53, 57], can be applied for one.

To establish boundedness of the measure μ_λ, we need the following auxiliary result.

Lemma 5.3. *Let $\mathbf{h}$ be a solution of (3.9), and $\mu_{\mathbf{h}}$ be an associated p-adic Gibbs measure. Then for the corresponding partition function $Z_n^{(\mathbf{h})}$ (see (3.6)) the following equality holds*

$$Z_{n+1}^{(\mathbf{h})} = A_{\mathbf{h},n} Z_n^{(\mathbf{h})}, \tag{5.4}$$

where $A_{\mathbf{h},n}$ will be defined below (see (5.7)).

Proof. Since $\mathbf{h}$ is a solution of (3.9), then we conclude that there is a constant $a_{\mathbf{h}}(x) \in \mathbb{Q}_p$ such that

$$\prod_{y\in S(x)} \sum_{j\in\Phi} \exp_p\{J\delta_{ij} + h_{j,y}\}\lambda(j) = a_{\mathbf{h}}(x) \exp_p\{h_{i,x}\} \tag{5.5}$$

for any $i \in \Phi$. From this one gets

$$\begin{aligned} \prod_{x\in W_n} \prod_{y\in S(x)} \sum_{j\in\Phi} \exp_p\{J\delta_{ij} + h_{j,y}\}\lambda(j) &= \prod_{x\in W_n} a_{\mathbf{h}}(x) \exp_p\{h_{i,x}\} \\ &= A_{\mathbf{h},n} \exp_p\Big\{ \sum_{x\in W_n} h_{i,x} \Big\}, \end{aligned} \tag{5.6}$$

where

$$A_{\mathbf{h},n} = \prod_{x\in W_n} a_{\mathbf{h}}(x). \tag{5.7}$$

By (3.5),(5.6) we have

$$\begin{aligned}
1 &= \sum_{\sigma\in\Omega_n}\sum_{\omega\in\Phi}\mu_{\mathbf{h}}^{(n+1)}(\sigma\vee\omega)\\
&= \sum_{\sigma\in\Omega_n}\sum_{\omega\in\Phi}\frac{1}{Z_{n+1}^{(\mathbf{h})}}\exp_p\left\{H(\sigma\vee\omega)+\sum_{x\in W_{n+1}}h_{\omega(x),x}\right\}\prod_{x\in V_n}\lambda(\sigma(x))\prod_{y\in W_{n+1}}\lambda(\omega(y))\\
&= \frac{1}{Z_{n+1}^{(\mathbf{h})}}\sum_{\sigma\in\Omega_n}\exp_p\{H(\sigma)\}\prod_{x\in V_n}\lambda(\sigma(x))\prod_{x\in W_n}\prod_{y\in S(x)}\sum_{j\in\Phi}\exp_p\{J\delta_{\sigma(x),j}+h_{j,y}\}\lambda(j)\\
&= \frac{A_{\mathbf{h},n}}{Z_{n+1}^{(\mathbf{h})}}\sum_{\sigma\in\Omega_n}\exp_p\left\{H(\sigma)+\sum_{x\in W_n}h_{\sigma(x),x}\right\}\prod_{x\in V_n}\lambda(\sigma(x))\\
&= \frac{A_{\mathbf{h},n}}{Z_{n+1}^{(\mathbf{h})}}Z_n^{(\mathbf{h})}
\end{aligned}$$

which implies the required relation. □

Now, as before, assume that condition (4.6) is satisfied. Then we know that the equation (3.9) has a unique translation invariant solution $\hat{\mathbf{h}}_\lambda=\{\hat{h}_{\lambda,n}\}$, therefore, $a_{\mathbf{h}}(x)$ does not depend on x, which will be denoted by a. Hence, from (5.5) one finds

$$a=\exp_p\{(k-1)h_{\lambda,0}\}(\lambda(0))^k\left(\theta+\sum_{j=1}^{\infty}\hat{h}_{\lambda,j}\right)^k. \tag{5.8}$$

The equalities (5.4) and (5.7) imply that

$$Z_{\lambda,n}=a^{|V_{n-1}|}, \tag{5.9}$$

where $Z_{\lambda,n}$ denotes the partition function of the measure μ_λ corresponding to the unique solution. Now we are ready to formulate a result.

Theorem 5.4. *Assume that* (4.6) *is satisfied. Then the p-adic Gibbs measure μ_λ is bounded.*

Proof. Take any $\sigma\in\Omega_{V_n}$. Then from (3.5) with (5.9), (4.6) one gets

$$\begin{aligned}
|\mu_\lambda(\sigma)|_p &= \frac{1}{|Z_{\lambda,n}|_p}\left|\exp_p\{H(\sigma)+\sum_{x\in W_n}h_{\lambda,\sigma(x)}\}\prod_{x\in V_n}\lambda(\sigma(x))\right|_p\\
&= \frac{1}{|\lambda(0)|_p^{k|V_{n-1}|}|\theta+\sum_{j=1}^{\infty}\hat{h}_{\lambda,j}|_p^k}\prod_{x\in V_n}|\lambda(\sigma(x))|_p\\
&= \frac{|\lambda(0)|_p^{|V_n|}}{|\lambda(0)|_p^{k|V_{n-1}|}}\prod_{x\in V_n}\left|\frac{\lambda(\sigma(x))}{\lambda(0)}\right|_p\\
&\le |\lambda(0)|_p^{|V_n|-k|V_{n-1}|},
\end{aligned} \tag{5.10}$$

here we have used that $|\theta+\sum_{j=1}^{\infty}\hat{h}_{\lambda,j}|_p=1$. It is known [9] that

$$|V_n|=1+\frac{k+1}{k-1}(k^n-1),$$

therefore,

$$|V_n|-k|V_{n-1}|=2$$

hence, (5.10) implies that μ_λ is bounded. □

Example 5.1. Assume that a solution of (3.9) is $\hat{h}_i = p^i$, $i \in \mathbb{N}$. In this case, one can see that

$$\sum_{j=1}^{\infty} \hat{h}_j = \frac{p}{1-p}.$$

Let us find the corresponding weight λ. Put $\lambda(0) = 1$, then from (3.9) one gets

$$\lambda(n) = p^n \left(\frac{p(1-\theta)+\theta}{(\theta-1)p^n(1-p)+1} \right)^k, \quad n \in \mathbb{N}$$

which evidently satisfies conditions (3.4) and (4.6). So, there is a unique bounded p-adic Gibbs measure on Ω.

6. Certain properties of the p-adic Gibbs measures

In this section without loss of generality we assume that for weights $\lambda(0) = 1$ and $h_0 = 0$.

Denote

$$\mathcal{W} = \{\{\lambda(i)\}_{i\in\Phi} \in \mathbb{Q}_p^{\Phi} : \ \lambda(0) = 1, \ |\lambda(i)|_p < 1, \ |\lambda(n)|_p \to 0 \text{ as } n \to \infty\}.$$

A norm of a weight $\lambda \in \mathcal{W}$ we define by $\|\lambda\|_{\mathcal{W}} = \max_{n\in\Phi}\{|\lambda(n)|_p\}$, therefore, it is clear that $\|\lambda\|_{\mathcal{W}} = 1$. We know from Theorems 5.2 and 5.4 that for every $\lambda \in \mathcal{W}$ there is a unique bounded p-adic Gibbs measure μ_λ, by denote $\mathcal{G}_P$ the set of such measures corresponding to the homogeneous Potts model (3.1). We endow $\mathcal{G}_P$ with a norm defined by

$$\|\mu\|_{\mathcal{G}} = \max_{\substack{\sigma\in\Omega_{V_n} \\ n\in\mathbb{N}}} |\mu(\sigma)|_p, \quad \mu \in \mathcal{G}_P. \tag{6.1}$$

One can ask: does the measure μ_λ depend on λ continuously? Next Theorem gives an answer to the question.

Theorem 6.1. *For every $\lambda, \kappa \in \mathcal{W}$ one has*

$$\|\mu_\lambda - \mu_\kappa\|_{\mathcal{G}} \le \|\lambda - \kappa\|_{\mathcal{W}}. \tag{6.2}$$

Hence, the correspondence $\lambda \mapsto \mu_\lambda$ is continuous.

Proof. Take any $\lambda, \kappa \in \mathcal{W}$. Let $\hat{\mathbf{h}}_\lambda = \{\hat{h}_{\lambda,i}\}_{i\in\mathbb{N}}$ and $\hat{\mathbf{h}}_\kappa = \{\hat{h}_{\kappa,i}\}_{i\in\mathbb{N}}$ be the corresponding solutions of (3.9)[4]. Denote $\mathbf{h}_\lambda = \{h_{\lambda,i} = \log_p \hat{h}_{\lambda,i}\}$, $\mathbf{h}_\kappa = \{h_{\kappa,i} = \log_p \hat{h}_{\kappa,i}\}$, which exist due to the proof of Theorem 5.2. Using (4.3) consider the difference

$$\begin{aligned} |\hat{h}_{\lambda,i} - \hat{h}_{\kappa,i}|_p &= |\lambda(i)F_i(\hat{\mathbf{h}}_\lambda;\theta) - \kappa(i)F_i(\hat{\mathbf{h}}_\kappa;\theta)|_p \\ &= |\lambda(i)(F_i(\hat{\mathbf{h}}_\lambda;\theta) - F_i(\hat{\mathbf{h}}_\kappa;\theta)) + (\lambda(i)-\kappa(i))F_i(\hat{\mathbf{h}}_\kappa;\theta)|_p \\ &\le \max\left\{|\lambda(i)|_p|F_i(\hat{\mathbf{h}}_\lambda;\theta) - F_i(\hat{\mathbf{h}}_\kappa;\theta)|, |\lambda(i)-\kappa(i)|\right\} \\ &\le \max\left\{|\lambda(i)|_p|\theta-1|_p\|\hat{\mathbf{h}}_\lambda - \hat{\mathbf{h}}_\kappa\|, |\lambda(i)-\kappa(i)|\right\}. \end{aligned} \tag{6.3}$$

Taking into account $|\lambda(i)|_p|\theta-1|_p\|\hat{\mathbf{h}}_\lambda - \hat{\mathbf{h}}_\kappa\| < \|\hat{\mathbf{h}}_\lambda - \hat{\mathbf{h}}_\kappa\|$ from (6.3) and non-Archimedeanity of the norm $\|\cdot\|$ one has

$$\|\hat{\mathbf{h}}_\lambda - \hat{\mathbf{h}}_\kappa\| = \|\lambda - \kappa\|_{\mathcal{W}}. \tag{6.4}$$

[4] Here we again recall that according to Theorems 4.3 and 5.2 such solutions exist.

Now take any $n \in \mathbb{N}$. Let us estimate the difference between the partition functions $Z_{\lambda,n}$ and $Z_{n,\kappa}$ (see (3.6)) of the measures μ_λ and μ_κ, respectively. Thanks to (5.9) with (5.8) and (4.2) one gets

$$\begin{aligned}
|Z_{\lambda,n} - Z_{\kappa,n}|_p &= \left|\left(\theta + \sum_{j=1}^{\infty} \hat{h}_{\lambda,j}\right)^k - \left(\theta + \sum_{j=1}^{\infty} \hat{h}_{\kappa,j}\right)^k\right|_p \\
&\leq \left|\sum_{j=1}^{\infty} \hat{h}_{\lambda,j} - \sum_{j=1}^{\infty} \hat{h}_{\lambda,j}\right|_p \\
&\leq \|\hat{\mathbf{h}}_\lambda - \hat{\mathbf{h}}_\kappa\| \qquad (6.5)
\end{aligned}$$

Hence, for any $\sigma \in \Omega_{V_n}$, from (3.5) and Lemma 4.2 using (6.4),(6.5) we obtain

$$\begin{aligned}
|\mu_\lambda(\sigma) - \mu_\kappa(\sigma)|_p &= \Bigg|\frac{1}{Z_{\lambda,n}} \exp_p\left\{\sum_{x\in W_n} h_{\lambda,\sigma(x)}\right\} \prod_{u\in V_n} \lambda(\sigma(u)) \\
&\quad - \frac{1}{Z_{\kappa,n}} \exp_p\left\{\sum_{x\in W_n} h_{\kappa,\sigma(x)}\right\} \prod_{u\in V_n} \kappa(\sigma(u))\Bigg|_p \\
&= \Bigg|\frac{1}{Z_{\lambda,n}} \prod_{x\in W_n} \hat{h}_{\lambda,\sigma(x)} \prod_{u\in V_{n-1}} \lambda(\sigma(u)) \\
&\quad - \frac{1}{Z_{\kappa,n}} \prod_{x\in W_n} \hat{h}_{\kappa,\sigma(x)} \prod_{u\in V_{n-1}} \kappa(\sigma(u))\Bigg|_p \\
&\leq \max_{\substack{x\in W_n \\ u\in V_{n-1}}} \Bigg\{\left|\frac{1}{Z_{\lambda,n}} - \frac{1}{Z_{\kappa,n}}\right|_p, |\hat{h}_{\lambda,\sigma(x)} - \hat{h}_{\kappa,\sigma(x)}|_p, \\
&\quad |\lambda(\sigma(u)) - \kappa(\sigma(u))|_p\Bigg\} \\
&\leq \max\left\{|Z_{\lambda,n} - Z_{\kappa,n}|_p, \|\hat{\mathbf{h}}_\lambda - \hat{\mathbf{h}}_\kappa\|, \|\lambda - \kappa\|_{\mathcal{W}}\right\} \\
&\leq \|\lambda - \kappa\|_{\mathcal{W}}, \qquad (6.6)
\end{aligned}$$

here we have used equalities $|Z_{\lambda,n}|_p = 1$, $|Z_{\kappa,n}|_p = 1$, which come from (5.9).

Due to the arbitrariness of n and σ we get the required relation (6.2). □

Now consider one limit theorem concerning μ_λ.

Let us fix a wight $\lambda \in \mathcal{W}$ such that $\lambda(i) \neq 0$ for all $i \in \mathbb{N}$. As before by $\hat{\mathbf{h}}_\lambda$ we denote a solution of (3.9), and, as before, $h_{\lambda,i} = \log_p \hat{h}_{\lambda,i}$, $i \in \mathbb{N}$.

Let us denote

$$A_n = \left\{\sigma \in \Omega_{V_n} : J \sum_{<x,y>\in V_n} \delta_{\sigma(x),\sigma(y)} + \sum_{x\in W_n} h_{\lambda,\sigma(x)} \equiv 0(mod\ p^n), \right\}. \qquad (6.7)$$

By $\lambda^{\otimes,n}$ denote the following measure defined on Ω_{V_n}

$$\lambda^{\otimes,n}(\sigma) = \frac{1}{Z_{\lambda,n}} \prod_{x\in V_n} \lambda(\sigma(x)), \quad \sigma \in \Omega_{V_n}. \qquad (6.8)$$

Theorem 6.2. *Let μ_λ be the p-adic Gibbs measure corresponding to the Potts model with a weight λ. Then one has*

$$\max_{\sigma\in A_n} \left|\frac{\mu_\lambda(\sigma)}{\lambda^{\otimes,n}(\sigma)} - 1\right|_p \to 0 \quad as \quad n \to \infty. \qquad (6.9)$$

Proof. Note that if $\sigma \in A_n$ then it means that

$$J \sum_{<x,y>\in V_n} \delta_{\sigma(x),\sigma(y)} + \sum_{x\in W_n} h_{\lambda,\sigma(x)} = Mp^n$$

for some $M \in \mathbb{Z}$. Therefore, the last equality with (6.8), (3.5) implies that

$$\begin{aligned} \left|\frac{\mu_\lambda(\sigma)}{\lambda^{\otimes,n}(\sigma)} - 1\right|_p &\leq \left|\exp_p\{Mp^n\} - 1\right|_p \\ &= |Mp^n|_p \\ &\leq p^{-n} \to 0 \ \text{ as } \ n \to \infty \end{aligned}$$

which proves the assertion. □

Remark 6.1. In the theory of Markov process it is important to know whether a given Markov measure or Markov field has some clustering (i.e. mixing) property. It is known [27] that, in the real case, when a Gibbs measure is unique, then it basically has that property. In a p-adic case, the situation is rather tricky (see [34, 35]) and still lots are not known. The proven Theorem shows that we are able to find a set A_n on which the p-adic measure μ_λ becomes product measure when n is large enough, which means μ_λ has a clustering property on A_n. The results in [32, 34] show that, in general, the clustering property does not hold for p-adic measures.

Remark 6.2. Note that in [35] the limit behaviour of sums of independent equally distributed random variables with respect to p-adic Bernoulli measure has been studied. The measure μ_λ defines a Markov process with infinite number of states, so the last theorem is the first step in an investigation of limit theorems for dependent processes in a p-adic context.

7. Conclusions

Investigations of physical models over p-adic field require a new kind of probability theory [31],[40]. Development of the p-adic probability theory gives a possibility to study p-adic statistical mechanics models. In the present paper we have studied countable state nearest-neighbor p-adic Potts models on a Cayley tree in the p-adic probability scheme. For the the model, we gave a construction of p-adic Gibbs measures which depends on weight λ. Such measures are natural and provide nontrivial concrete examples of p-adic Markov processes with countable state space (see [39]). We have shown that under some condition on weights for the model, the absence of a phase transition by studying an infinite-dimensional recursion equation. Such results are unknown for the real counterparts of the considered models (see [24, 26]). It turned out that the condition does not depend on values of the prime p, therefore an analogous fact is not true when the number of spins is finite [49, 50]. It has been proven that p-adic Gibbs measure for homogeneous Potts model is unique. We also established boundedness such a measure. Continuous dependence the measure on weights was also proven. We even obtain one limit theorem for such a measure. This is the first step in an investigation on limit theorems for dependent processes in a p-adic context.

Acknowledgement

The author F.M. thanks the FCT (Portugal) grant SFRH/BPD/17419/2004. He is also grateful to Prof. A.Yu. Khrennikov at Växjö University for kind hospitality. The third named author J.F.F.M. thanks DYSONET-project for partial support. Finally, the authors also would like to thank to the referee for his useful suggestions which allowed us to improve the text of the paper.

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